New results from relativistic geodesy

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CENTER OF
APPLIED SPACE TECHNOLOGY
AND MICROGRAVITY
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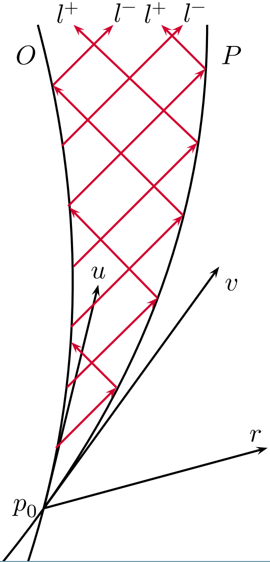
Basic notions defined by General Relativity

based on (freely falling) particles and light propagation (unique)

- standard clock
  GR tells you what a good clock is
  (Perlick, GRG 1987)
  atomic clocks are good clocks



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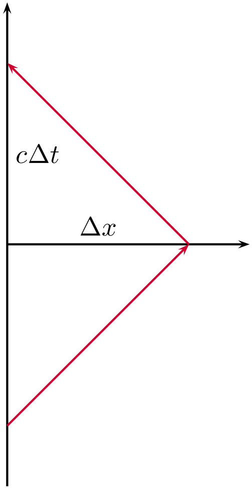
$$\Delta x = c\,\Delta\tau$$



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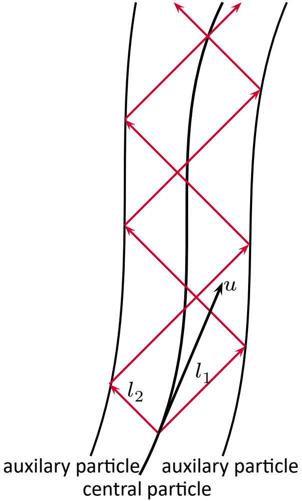
- rotation
  GR tells you what rotation is (Pirani, BAP 1965)
  gyroscopes show this rotation



Model of the Earth

Theorem

If for a body holds:

- it is rigid (no shear, no expansion)
- the rotation is constant, does not precess: $P_u D_u \omega = 0$ and
- the local acceleration rotates with the rigid body $P_u D_u a = \omega \cdot a$

then the body is stationary: $\exists \xi \sim u$ with $\mathscr{L}_\xi g = 0$ (Salzmann & Taub 1964, Ehlers 1961)

- conditions are fulfilled for the Earth with good accuracy
- involved velocities and masses are small $\Rightarrow$ any time-dependence can be included *adiabatically* with very high precision

the Earth is a Killing congruence ξ giving a stationary space-time $\mathscr{L}_\xi g_{\mu\nu} = 0$

Stationary space-time

Stationarity, model of Earth's gravity: time-like Killing congruence ξ

any time-like Killing vector field ξ has two characteristics

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- norm $e^{2\phi} = g_{\mu\nu}\xi^\mu\xi^\nu$ related to
  - acceleration
  - gravitational redshift
  - qm phase shift

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- twist $\varpi^\mu := \epsilon^{\mu\nu\rho\sigma}\xi_\nu\partial_\rho\xi_\sigma$ related to
  - Sagnac effect
  - Schiff effect
  - qm spin-rotation coupling

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Theorem (e.g. Israel & Wilson, JMP 1972, Simon & Beig, JMP 1982)

- for the twist one has $\partial_{[\mu}\varpi_{\nu]} = -\epsilon_{\mu\nu\rho\sigma}\xi^\rho R^\sigma{}_\tau\xi^\tau$
- if vacuum Einstein equations are fulfilled $\partial_{[\mu}\varpi_{\nu]} = 0$ and $\exists\ \varpi : \varpi_\mu = \partial_\mu\varpi$

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two potentials: ϕ gravitoelectric (redshift/acceleration) , ϖ gravitomagnetic (rotation)

The full gravitational field

in adapted coordinates one has for the full metric

$$ds^2 = e^{2\phi}\left(dt + \sigma_i dx^i\right)^2 - e^{-2\phi}\gamma_{ij}dx^i dx^j \qquad i,j,k,\ldots = 1,2,3 \qquad \varpi^i = e^{4\phi}\epsilon^{ijk}\partial_j\sigma_k$$

mass and angular momentum potentials

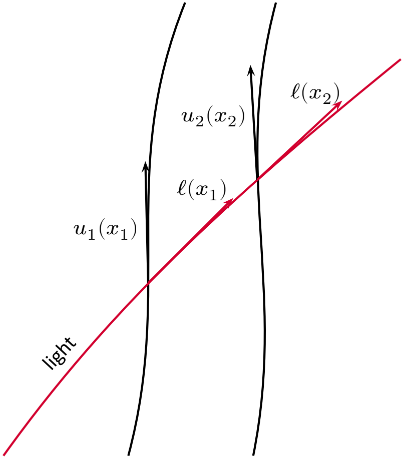
$$\Phi_M := \frac{1}{4}e^{-2\phi}\left(e^{4\phi} + \varpi^2 - 1\right), \qquad \Phi_J := \frac{1}{2}e^{-2\phi}\varpi$$

multipole expansion (Hansen, JMP 1974, Simon & Beig, JMP 1983)

$$\Phi_M = \sum_{l=0}^{m-1} \frac{E_{i_1\cdots i_l}x^{i_1}\cdots x^{i_l}}{l!\,r^{2l+1}} + \mathscr{O}(r^{-(m+1)}) \qquad \Phi_J = \sum_{l=0}^{m-1} \frac{F_{i_1\cdots i_l}x^{i_1}\cdots x^{i_l}}{l!\,r^{2l+1}} + \mathscr{O}(r^{-(m+1)})$$

$$\gamma_{ij} = \delta_{ij} + \sum_{l=2}^{m}\left(\frac{x^i x^j A_{a_1\cdots a_{l-2}}x^{a_1}\cdots x^{a_{l-2}}}{r^{2l}} + \frac{\delta^{ij}B_{a_1\cdots a_{l-2}}x^{a_1}\cdots x^{a_{l-2}}}{r^{2l-2}} + \frac{x_{(i}C_{j)a_1\cdots a_{l-3}}x^{a_1}\cdots x^{a_{l-3}}}{r^{2l-2}} \right.$$
$$\left. + \frac{D_{ija_1\cdots a_{l-2}}x^{a_1}\cdots x^{a_{l-4}}}{r^{2l-4}}\right) + \ldots$$

Clock comparison within GR



- light ray given by trajectory with tangent $\ell(\lambda)$ obeying $g(\ell, \ell) = 0$
  light is unuique
- observer with standard clock given by trajectory with tangent $u(\tau)$ obeying $g(u, u) = 1$

Definition: measured frequency

$$\nu = g_{\mu\nu} \ell^\mu u^\nu = k_\mu u^\mu \quad \text{with} \quad k_\mu = g_{\mu\nu} \ell^\nu$$

clock comparison (z is called "redshift")

$$1 + z := \frac{\nu_1}{\nu_2} = \frac{k_\mu(x_1) u^\mu(x_1)}{k_\mu(x_2) u^\mu(x_2)}$$

depends on the two observer's position as well as state of motion (velocity)

Clock comparison within GR

- position effects: redshift
- motional effects: time delay (also aberration)

stationary gravitational field and stationary observer (observer at rest): gravitational redshift, access to full potential difference ($\overset{*}{=}$ in adapted coordinate system)

$$\frac{\nu_2}{\nu_1} = \frac{k(u_2)}{k(u_1)} = \frac{e^{\phi_2}}{e^{\phi_1}} \overset{*}{=} \sqrt{\frac{g_{tt}(r_2)}{g_{tt}(r_1)}} \overset{*}{\approx} 1 - \frac{U(x_1) - U(x_2)}{c^2}$$

same position but different velocities: Doppler effect and time dilation

$$\frac{\nu_2}{\nu_1} = \frac{k_\mu(x)u^\mu(x)}{k_\nu(x)v^\nu(x)} = \sqrt{1-v^2}\,(1 \pm v) \qquad \Leftrightarrow \qquad T' = \frac{1}{\sqrt{1-v^2}}T$$

radial motion

important on Earth: light may propagate along optical fibers (Philipp et al, PRD 2017)

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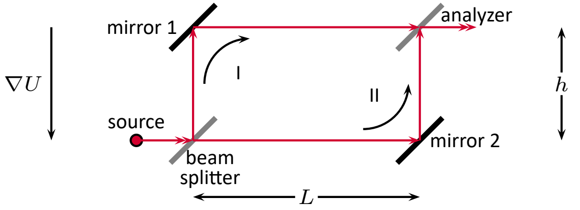
Interferometry

intensity and phase shift for stationary interferometer (Audretsch, C.L., JPA 1982, Kagramanova, Kunz, C.L., CQG 2008)

$$I = 2\,(1 + \cos \Delta\Phi)\, I_0 \qquad \text{with} \qquad \Delta\Phi = \oint p\,, \quad p = -dS$$

phase shift

$$\begin{aligned} \Delta\Phi &= E \int e^{-\phi}\omega + \oint P_u p \qquad \text{with} \qquad E = p(\xi) \\ &= E \int e^{-\phi}\omega + \oint \sqrt{E^2 e^{-2\phi} - m^2} n \\ &= E e^{-\phi}\omega \cdot \Sigma + \Delta\sqrt{E^2 e^{-2\phi} - m^2} L \\ &\approx E e^{-\phi}\omega \cdot \Sigma - \frac{\Delta\phi}{p_0} L \end{aligned}$$



Sagnac effect + full potential difference

Acceleration

for stationary apparatus (at rest)

from $u^\mu = e^{-\phi}\xi^\mu$ we obtain

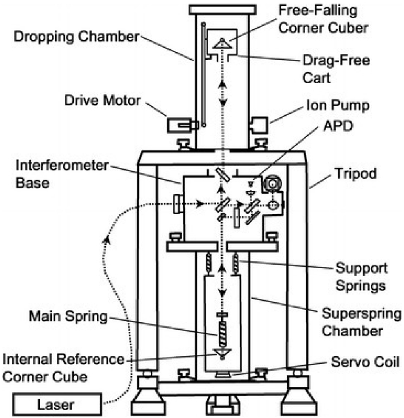
$$a_\mu = u^\rho D_\rho u_\mu = -\partial_\mu \phi$$

and

$$\begin{aligned} (P_u)^\mu_\rho u^\nu D_\nu a^\rho &= \omega^\mu{}_\nu a^\nu \\ 0 &= u^\mu \partial_\mu \phi \end{aligned}$$

measurable with falling corner cubes

access to gradient of potential

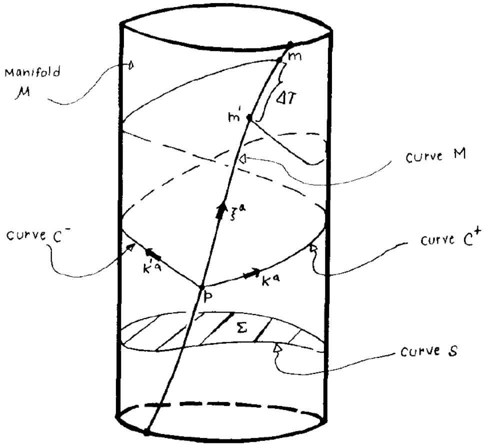


Measuring ϖ

the twist ϖ^μ can be measured with

- Sagnac effect (interferometry with ringlasers, Ashtekar & Magnon 1975)

$$\Delta t = \int_\Sigma e^{-\phi} \epsilon_{\rho\sigma\mu\nu} \xi^\sigma \varpi^\rho dS^{\mu\nu}$$



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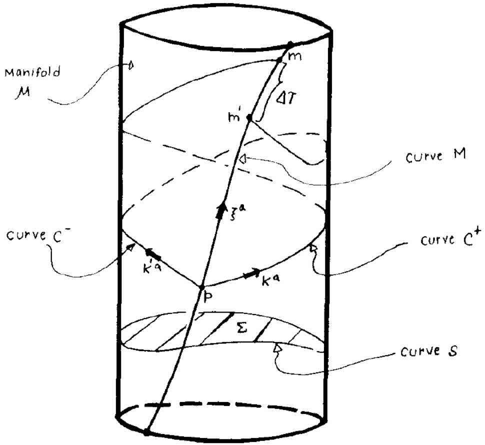

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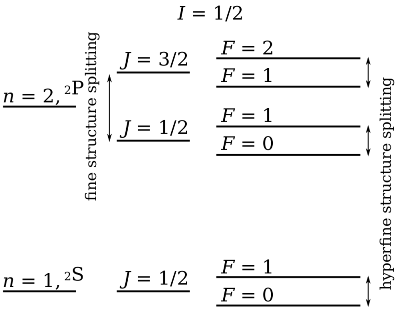
$$\Delta E = \vec{\omega} \cdot \vec{J}$$



Wikipedia

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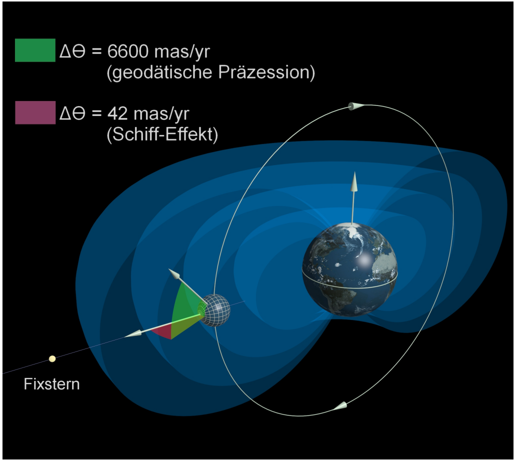
- spin precession (Zimbres, CQG 2014, also Rindler & Perlick, GRG 1990)

$$u^\nu D_\nu S^\mu = e^{-\phi} \varpi^\mu{}_\nu S^\nu$$

mission GP-B (Everitt et al, PRL 2012), proposal HYPER



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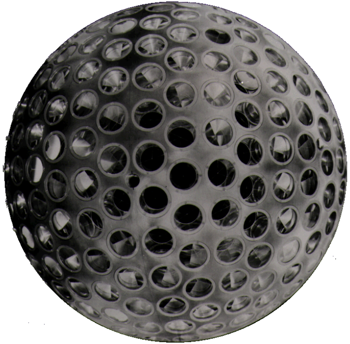
- Lense-Thirring (Ciufolini et al, EPJc 2019) and gravitomagnetic clock effect (proposal, Hackmann, C.L. PRD 2014)



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Measuring all degrees of freedom

degrees of freedom: ϕ, ϖ, γ_{ij} in $ds^2 = e^{2\phi}\left(dt + \varpi_i dx^i\right)^2 - e^{-2\phi}\gamma_{ij}dx^i dx^j$

- measurement of γ_{ij} requires non-stationary observers
- moving clocks, moving atom interferometers, moving gravimeters and gradiometers in space

general calculation scheme for geodesy

- given some multipole expansion $A_{i_1\cdots i_m}$, $B_{i_1\cdots i_m}$, $C_{i_1\cdots i_m}$, $D_{i_1\cdots i_m}$, $E_{i_1\cdots i_m}$ and $F_{i_1\cdots i_m}$
- determine from that $\Phi_M^{(m)}$, $\Phi_J^{(m)}$, and $\gamma_{ij}^{(m)}$
- in adiabatic time-dependent situations: mulipole parameters depend on time
- calculate metric and geodesic equation
- using this metric:
  - GRACE constellation and distance measurement
  - GOCE constellation
  - clock constellations
  - atom interferometers in space

this is one plan for the next few years ...

The two potentials for stationary axially symmetric space-times

analytical examples, toy models: stationary axially symmetric space-times

$$ds^2 = g_{tt}(r,\vartheta)dt^2 + g_{rr}(r,\vartheta)dr^2 + g_{\vartheta\vartheta}(r,\vartheta)d\vartheta^2 + g_{\varphi\varphi}(r,\vartheta)d\varphi^2 + 2g_{t\varphi}(r,\vartheta)dtd\varphi$$

gravitoelectric potential (with rotation)

$$e^{\phi_{\text{rot}}} = g_{\mu\nu}\left(\xi^\mu + \Omega\eta^\mu\right)\left(\xi^\nu + \Omega\eta^\nu\right) = g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi}$$

twist potential, gravitomagnetic potential (with rotation)

$$\begin{aligned}\varpi_\mu^{\text{rot}} &= g_{\mu\nu}\epsilon^{\nu\rho\sigma\tau}\left(\xi_\rho + \Omega\eta_\rho\right)\partial_\sigma\left(\xi_\tau + \Omega\eta_\tau\right)\\ &= \frac{1}{\sqrt{-g}}\begin{pmatrix} -g_{rr}\left(g_{tt}\partial_\vartheta g_{t\varphi} - g_{t\varphi}\partial_\vartheta g_{tt} + \Omega\left(g_{tt}\partial_\vartheta g_{\varphi\varphi} - g_{\varphi\varphi}\partial_\vartheta g_{tt}\right) + \Omega^2\left(g_{t\varphi}\partial_\vartheta g_{\varphi\varphi} - g_{\varphi\varphi}\partial_\vartheta g_{t\varphi}\right)\right)\\ g_{\vartheta\vartheta}\left(g_{tt}\partial_r g_{t\varphi} - g_{t\varphi}\partial_r g_{tt} + \Omega\left(g_{tt}\partial_r g_{\varphi\varphi} - g_{\varphi\varphi}\partial_r g_{tt}\right) + \Omega^2\left(g_{t\varphi}\partial_r g_{\varphi\varphi} - g_{\varphi\varphi}\partial_r g_{t\varphi}\right)\right)\end{pmatrix}\end{aligned}$$

with $g := \left(g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2\right)g_{rr}g_{\vartheta\vartheta}$
can integrate for gravitomagnetic potential ϖ for Schwarzschild, Kerr, Kerr-NUT, ...

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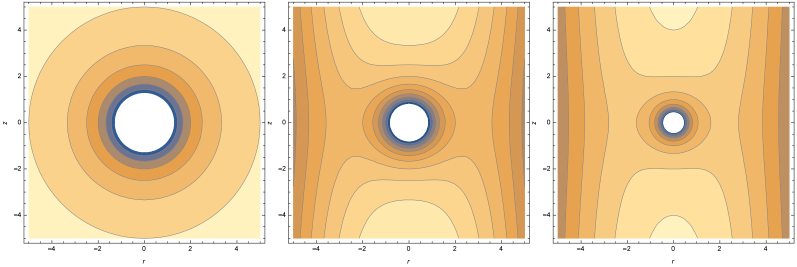
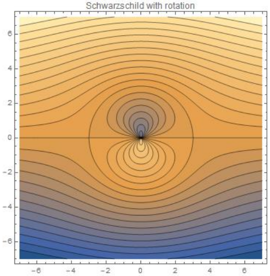
Gravitoelectric and -magnetic potentials for Schwarzschild

gravitoelectric potential



gravitomagnetic potential



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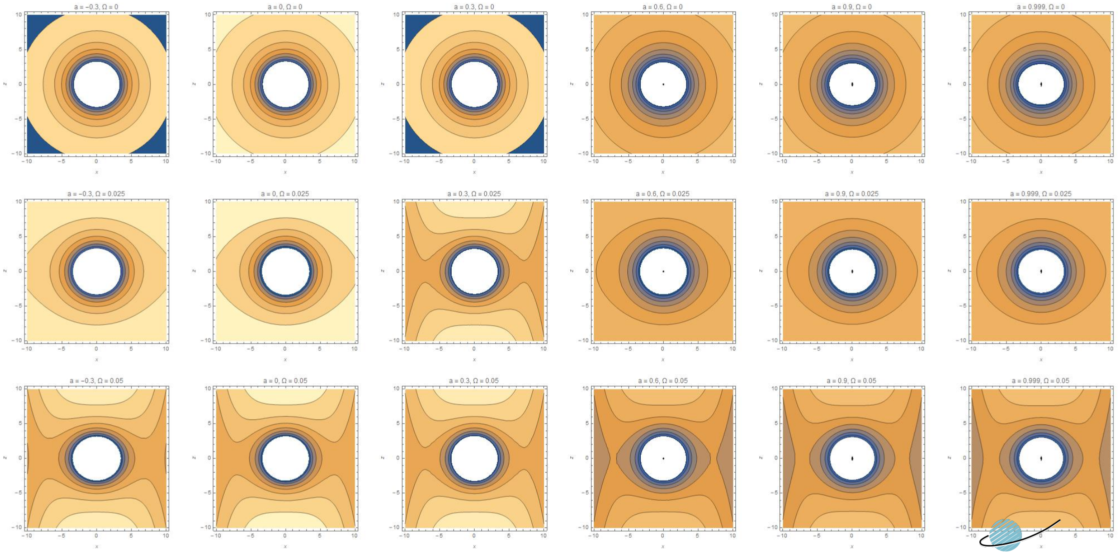
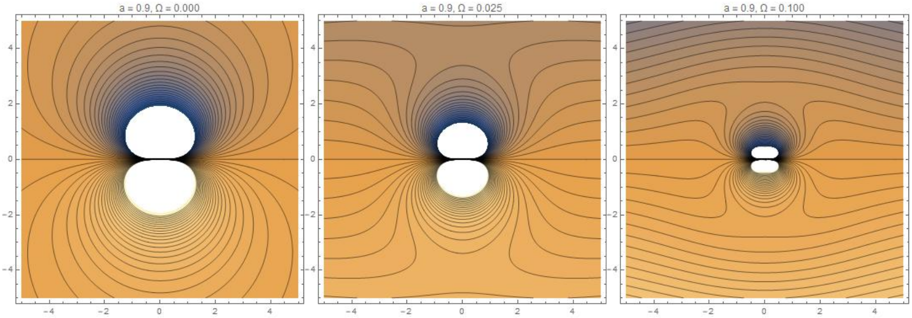
Gravitoelectric potential for Kerr



Gravitomagnetic potential for Kerr

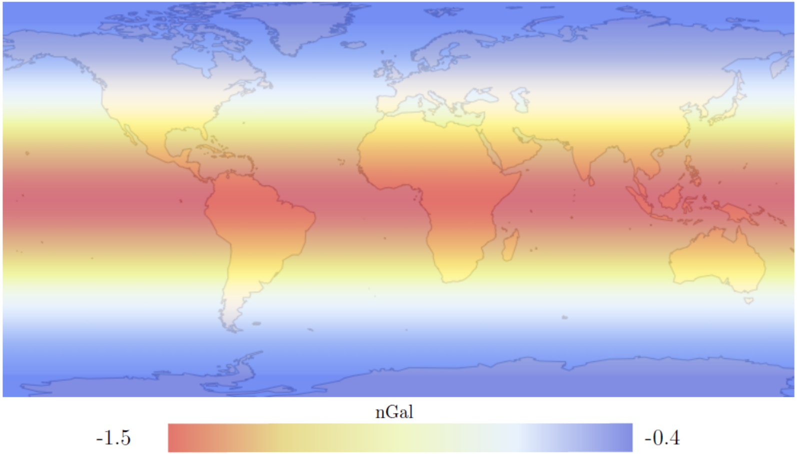


Further metrics

- Kerr-NUT metric
- q metric (general relativistic quadrupole)
- rotating q metric

Relativistic geoid: comparison

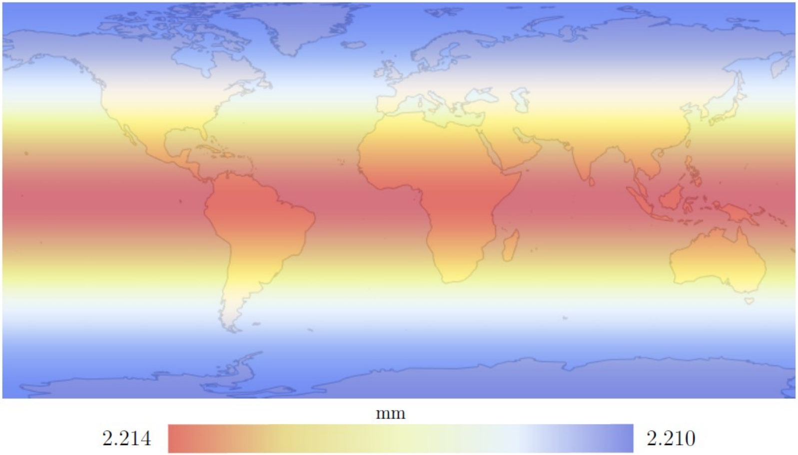
comparison of acceleration for Schwarzschild and Kerr, for same mass



Philipp, Hackmann, C.L., Müller, PRD 2020

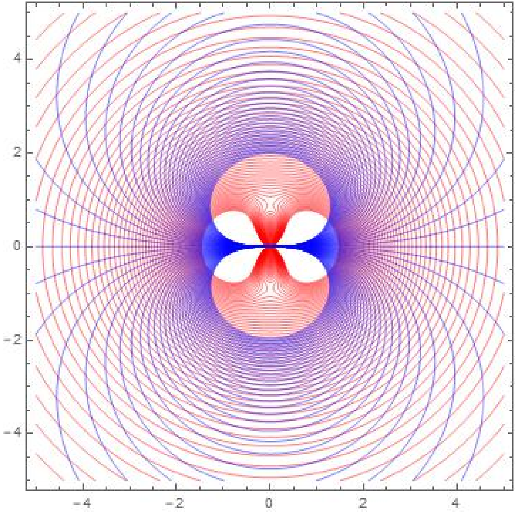
Relativistic geoid: comparison

isometric comparison of location of pN and Newtonian geoid, for axially symmetric quadrupole



Philipp, Hackmann, C.L., Müller, PRD 2020

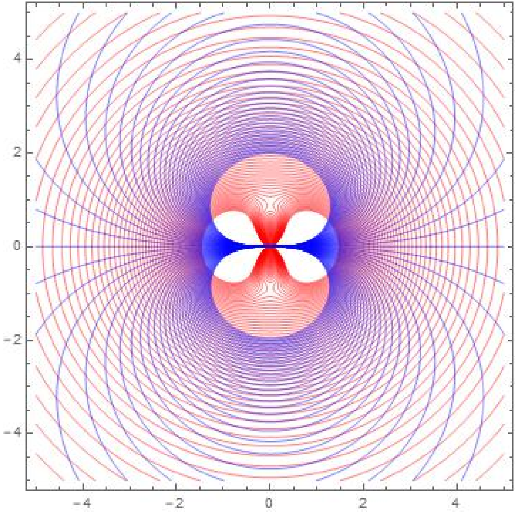
Both potentials for Kerr



measurement

- Kerr gravitational, acceleration, gravito-electric potential
  - with falling corner cubes we observe the gradient of the gravitational potential ϕ
  - with quantum devices (clocks, atom interferometers) we have direct access to the full potential
- Kerr twist, rotation, gravito-magnetic potential
  - with interferometers or spinning tops we can observe the gradient of ϖ
  - is there a possibility to have direct access to the twist potential?

Both potentials for Kerr



general physical interpretation

- twist related to axially symmetric spin multipole moments
- relevance for geodesy: mass fluxes

operational interpretation

- **gravito-electric potential**
  $\leftrightarrow$ gravitational definition of **height**
  sensitive to mass
- **gravito-magnetic potential**
  $\leftrightarrow$ gravitational definition of **latitude**
  sensitive to mass flux