Model development for the study of temporal networks of optical-atomic clocks

Miltiadis Chatzinikos and Pacôme Delva

SYRTE, Observatoire de Paris-PSL, Sorbonne Université, CNRS, LNE

Relativistic Geodesy with Clocks

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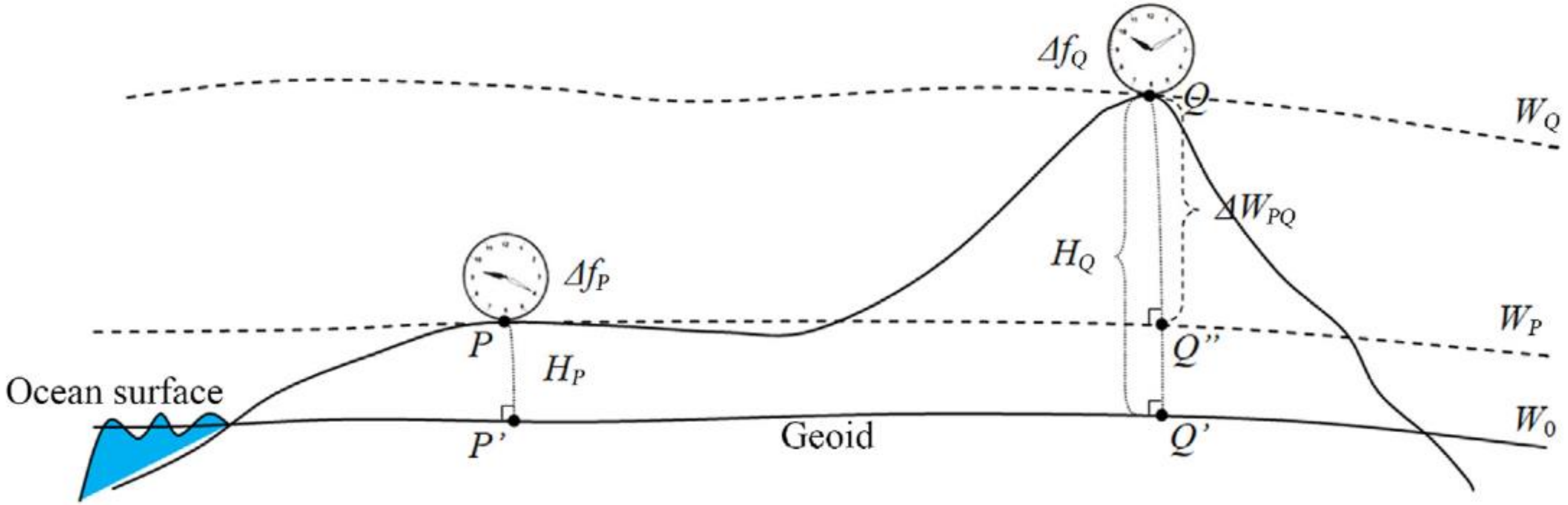
IAG QuGe Working Group 3

Fundamental observation equation

$$\frac{f_Q - f_P}{f} = -\frac{W_Q - W_P}{c^2}$$

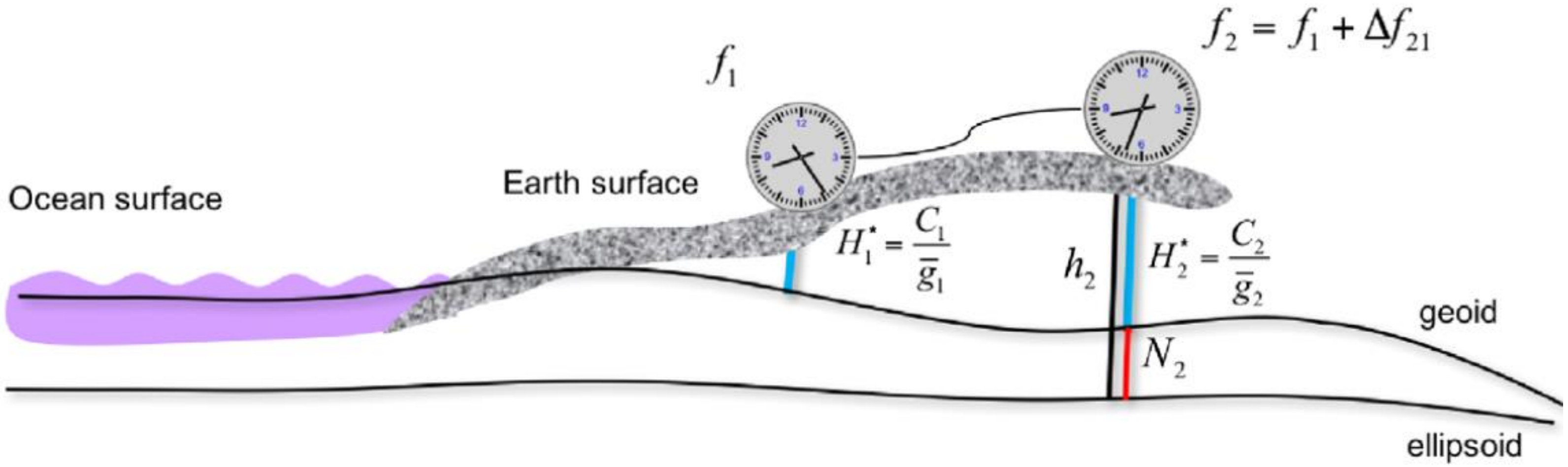
→ The relative gravity frequency shift equals the gravity potential difference between points P and Q



Chronometric Geodesy

$$\Delta H_{21}^* = H_2^* - H_1^* = H_1^* \frac{\Delta \bar{g}_{12}}{\bar{g}_2} + \frac{C_2 - C_1}{\bar{g}_2}$$
$$= H_1^* \frac{\Delta \bar{g}_{12}}{\bar{g}_2} - \frac{c^2}{\bar{g}_2} \frac{\Delta f_{21}}{f_1},$$

→ The difference between two orthometric heights is proportional to the relative gravity frequency shift and the gravity measurements



High Performance Clocks and Gravity Field Determination

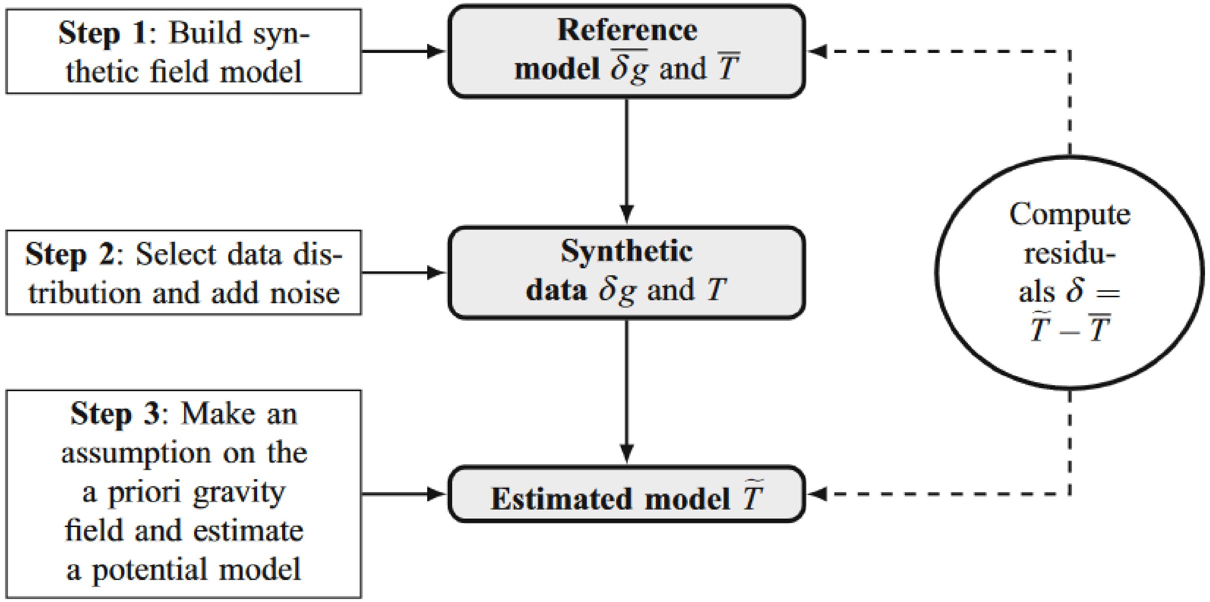
J. Müller · D. Dirkx · S.M. Kopeikin · G. Lion · I. Panet · G. Petit · P.N.A.M. Visser

What have been done in the past…

Determination of a high spatial resolution geopotential model using atomic clock comparisons

G. Lion[1,2] · I. Panet[2] · P. Wolf[1] · C. Guerlin[1,3] · S. Bize[1] · P. Delva[1]

Numerical approach strategy to evaluate the contribution of atomic clocks to estimate the geopotential model



Planar least squares collocation

Forsberg 1987: Logarithmic covariance model

$$\widetilde{T}_P = \mathbf{C}^{\mathsf{T}}_{T_P,l} \cdot \mathbf{C}^{-1}_{n,n} \cdot \mathbf{l}$$

$$\mathbf{C}_{n,n} = \mathbf{C}_{l,l} + \omega\, \mathbf{C}_{\epsilon,\epsilon}$$

$$\mathbf{C}_{\epsilon,\epsilon} = \begin{bmatrix} \mathbf{I}_p \cdot \sigma_T^2 & 0 \\ 0 & \mathbf{I}_q \cdot \sigma_{\delta g}^2 \end{bmatrix}$$

What have been done in the past…

Determination of high spatial resolution geopotential model

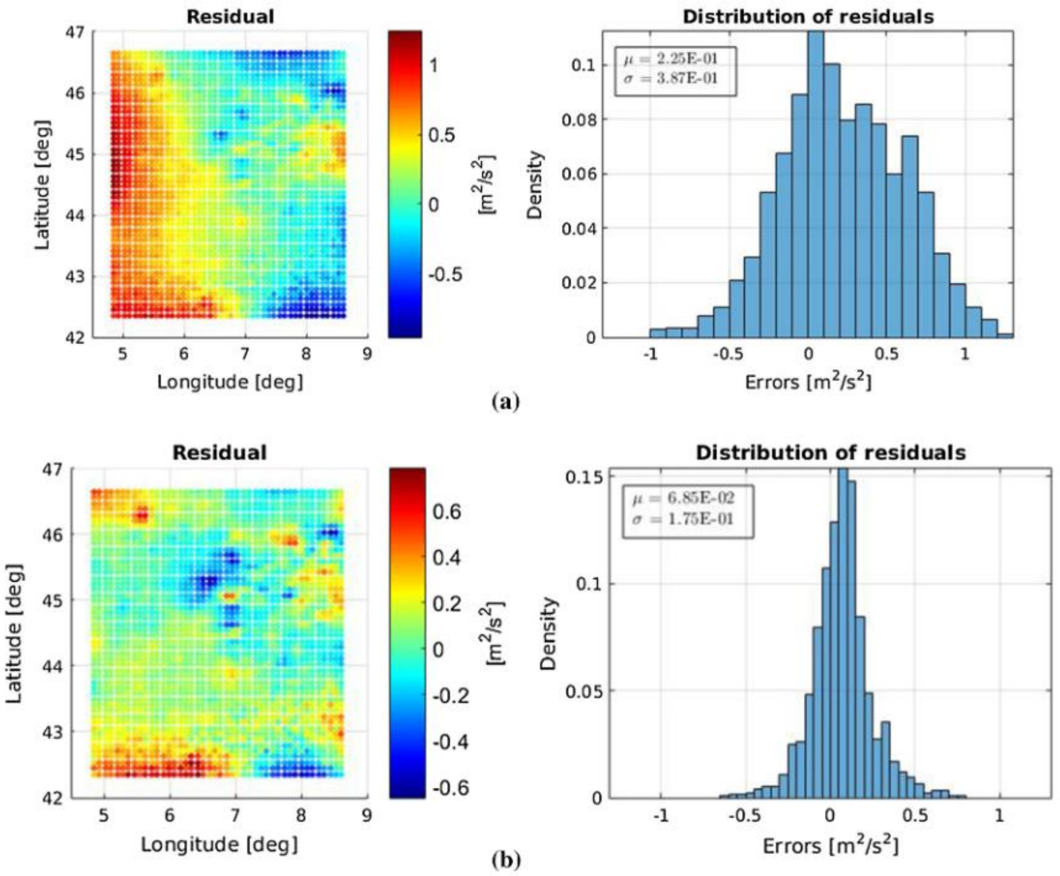


Fig. 10 Accuracy of the disturbing potential T reconstruction on a regular 10-km step grid in Alps, obtained by comparing the reference model and the reconstructed one. In **a**, the estimation is realized from the 4959 gravimetric data δg only and in **b** by adding 32 potential data T to the gravity data. **a** Without clock data, **b** With clock data

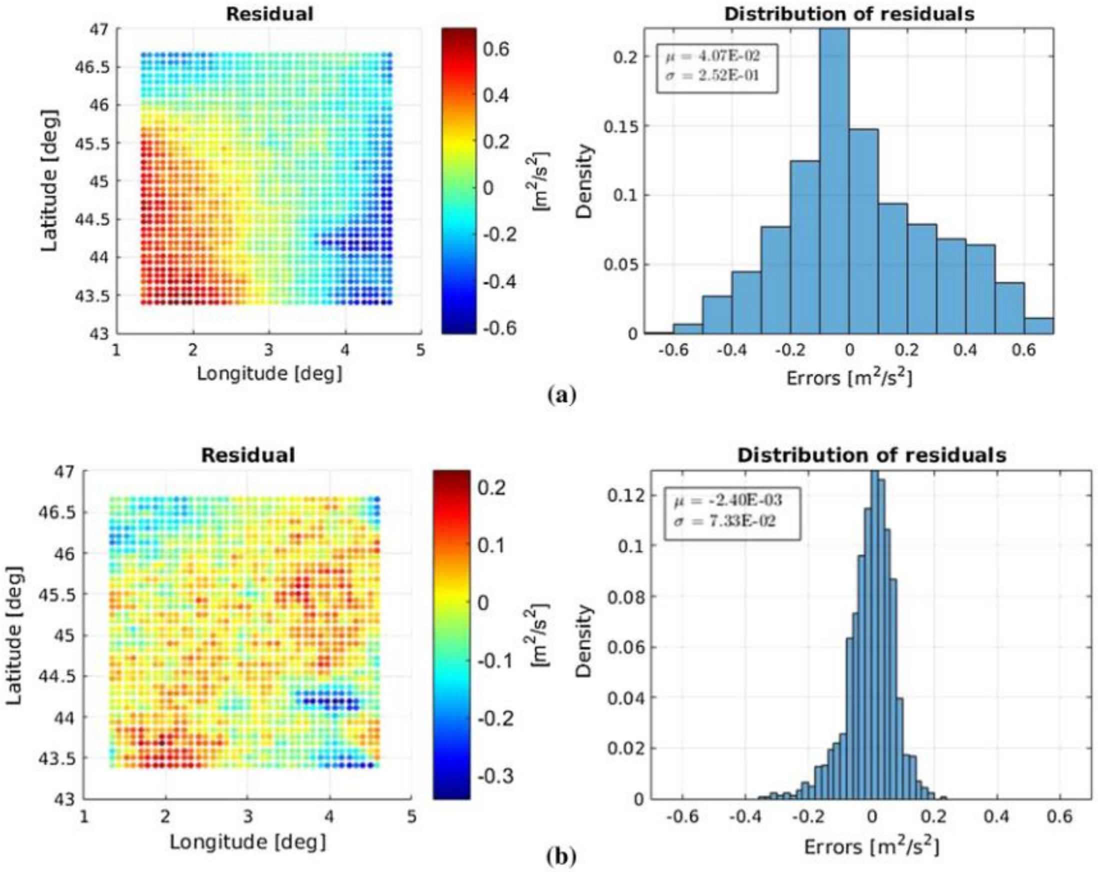


Fig. 9 Accuracy of the disturbing potential T reconstruction on a regular 10-km step grid in Massif Central, obtained by comparing the reference model and the reconstructed one. In **a**, the estimation is realized from the 4374 gravimetric data δg only and in **b** by adding 33 potential data T to the gravity data. **a** Without clock data, **b** With clock data

What have been done in the past…

Clock networks for height system unification: a simulation study

Hu Wu,[1] Jürgen Müller[1] and Claus Lämmerzahl[2]
[1]Institut für Erdmessung (IfE), Leibniz Universität Hannover, Schneiderberg 50, 30167 Hannover, Germany. E-mail: wuhu@ife.uni-hannover.de
[2]Zentrum für Angewandte Raumfahrttechnologie und Mikrogravitation (ZARM), Universität Bremen, Am Fallturm 2, 28359 Bremen, Germany

$$H_i^L = \frac{C_i^U}{\overline{\gamma}_i} + a^L \Delta X_i^L + b^L \Delta Y_i^L + c^L$$ → Functional model for normal heights in a local system

$$\Delta W_{ij} = W_i^U - W_j^U = -(C_i^U - C_j^U)$$ → Functional model for the clock base data

$$\begin{bmatrix} H \\ \Delta W \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ → Application of Least-squares adjustment for the estimation of solution

Clock networks for height system unification

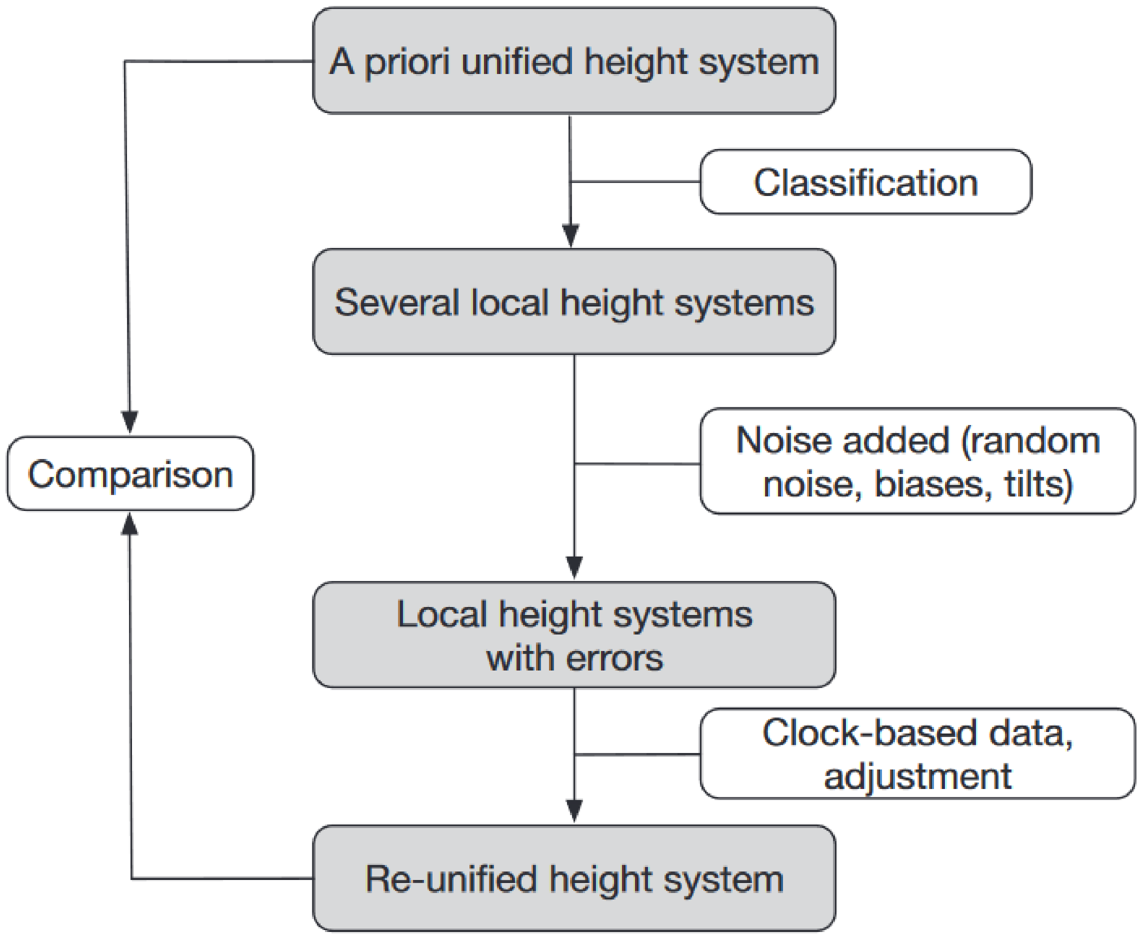
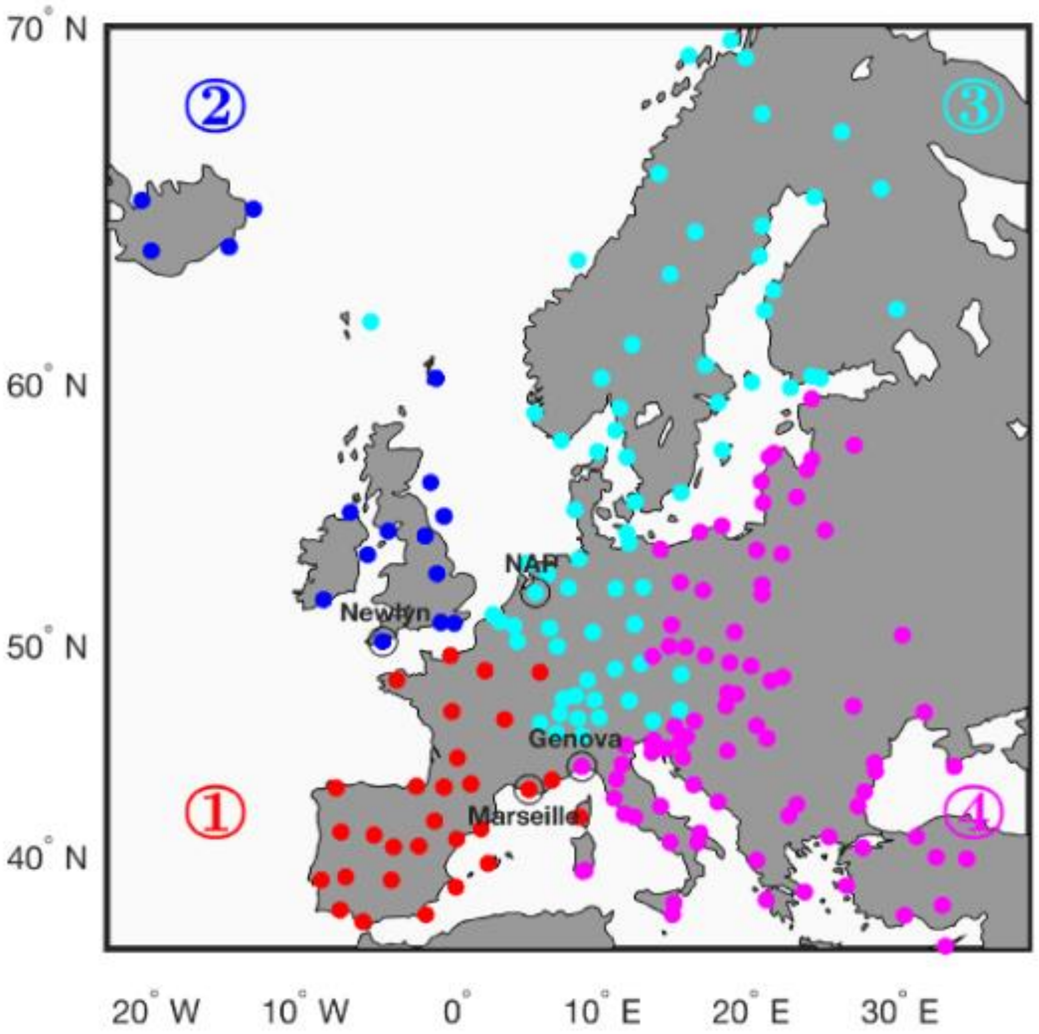
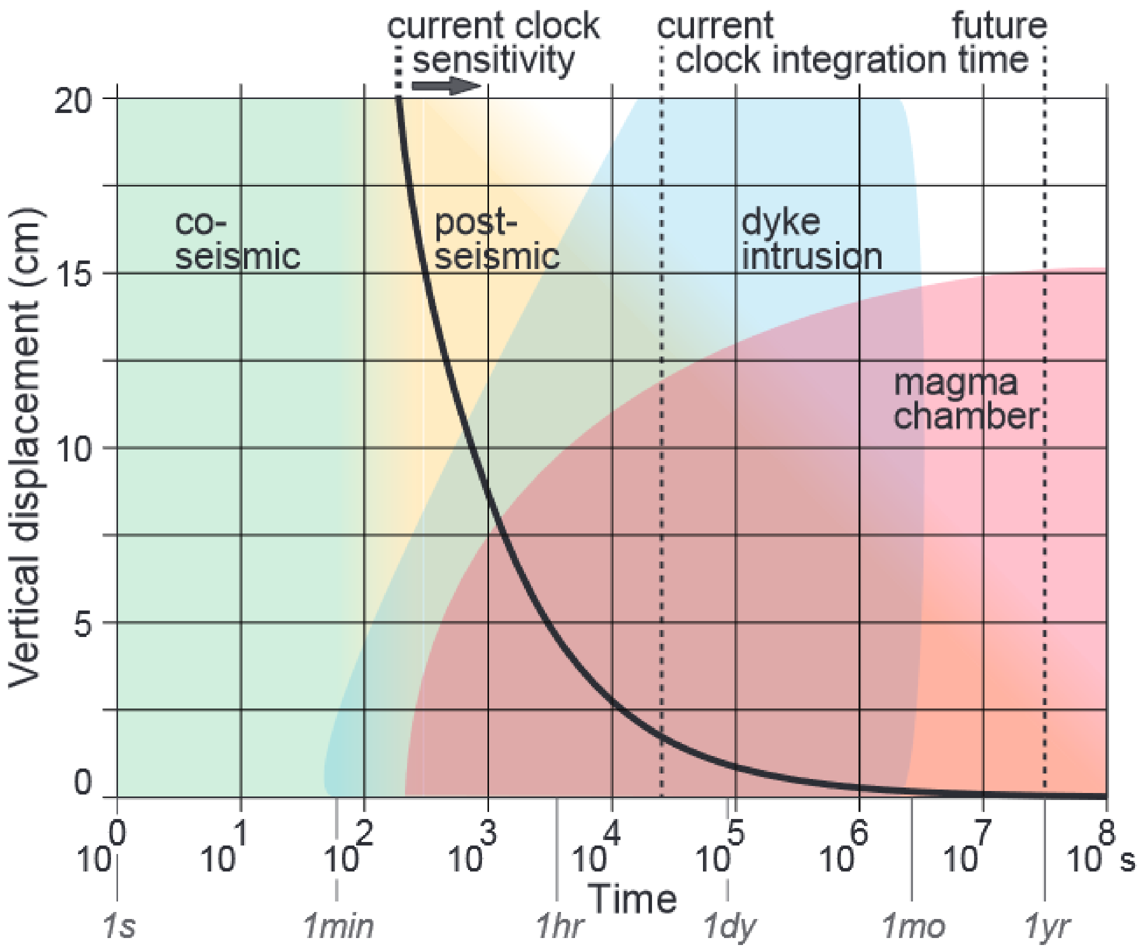


Figure 5. The scheme of the simulator.



What have been done in the past...



Atomic clocks as tools to monitor vertical surface motion

Ruxandra Bondarescu[1], Andreas Schärer[1], Andrew Lundgren[2]
György Hetényi[3,4], Nicolas Houlié[4,5], Philippe Jetzer[1], Mihai Bondarescu[6,7]

Phenomena that could be monitored with optical clock networks

$$\Delta f/f \sim 3 \times 10^{-16}/\sqrt{\tau/\mathrm{sec}}$$

$$\Delta f/f \sim \sigma_{\mathrm{tomorrow}} = 10^{-17}/\sqrt{\tau/\mathrm{sec}}$$

Current Objects

- Mathematical model for the analysis of products derived from the processing of clock measerements

- Study of two main different cases either via a deterministic approach or a stochastic one:
    - Spatial model
    - Spatial-temporal model

- Links between clock measurements/products and measurements/products derived from other geodetic instruments (e.g. GNSS, land gravimeters, etc)

- Development of a module

Functional Model

The general functional model:

$$\psi = f(\boldsymbol{x}_i, \boldsymbol{x}_j, W(\boldsymbol{x}_i), W(\boldsymbol{x}_j), g(\boldsymbol{x}_i), g(\boldsymbol{x}_j))$$

$\boldsymbol{x}$: position parameters

$\boldsymbol{W}(\boldsymbol{x})$: Gravity potential

$\boldsymbol{g}(\boldsymbol{x})$: Gravity measurements

Linearization of the model:

$$\psi = \psi^0 + \frac{\partial f}{\partial x_i^0}(x_i - x_i^0) + \frac{\partial f}{\partial x_j^0}(x_j - x_j^0) + \frac{\partial f}{\partial [W(x^0)]}[W(x) - U(x^0)] + \frac{\partial f}{\partial [g(x^0)]}[g(x) - \gamma(x^0)]$$

U: Normal gravity potential

γ: Normal gravity

x^0: approximate position

Observation Model

$$\boldsymbol{w} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{s} + \boldsymbol{V}$$

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{w}_n \end{bmatrix} \longrightarrow$$

Different group of observations
- Geodetic measurements
- Gravity measurements
- Clock measurements
- …

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} \longrightarrow$$

Position (unknown) parameters
- coordinates
- height
- …

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix} \longrightarrow$$

Residuals/Noise referred to the different group of observations

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \vdots \\ \boldsymbol{s}_p \end{bmatrix} \longrightarrow$$

Signals (unknown) parameters
- Gravity
- Gravitational potential
- Gravity anomaly
- …

Observation Model

$$\boldsymbol{w} = \boldsymbol{Ax} + \boldsymbol{Gs} + \boldsymbol{V}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_m \end{bmatrix} \longrightarrow$$ Jacobian matrix referred to the position parameters (**x**)

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{G}_1 \\ \boldsymbol{G}_2 \\ \vdots \\ \boldsymbol{G}_p \end{bmatrix} \longrightarrow$$ Jacobian matrix referred to the signals (**s**)

Least squares adjustment

$$\boldsymbol{V}^T\boldsymbol{PV} \rightarrow \boldsymbol{v}_1^T\boldsymbol{P}_1\boldsymbol{v}_1 + \boldsymbol{v}_2^T\boldsymbol{P}_2\boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n^T\boldsymbol{P}_n\boldsymbol{v}_n \rightarrow minimum$$

Least-Squares Adjustment - Spatial approach A

Deterministic approach of position and signal parameters:

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} + \boldsymbol{V}$$

$$\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{PA} & \boldsymbol{A}^T\boldsymbol{PG} \\ \boldsymbol{G}^T\boldsymbol{PA} & \boldsymbol{G}^T\boldsymbol{PG} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}} \\ \widehat{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{Pw} \\ \boldsymbol{G}^T\boldsymbol{Pw} \end{bmatrix}$$

$$\longrightarrow$$

$$\boldsymbol{V}^T\boldsymbol{PV} \rightarrow minimum$$

$$\begin{bmatrix} \boldsymbol{N}_x & \boldsymbol{N}_{xs} \\ \boldsymbol{N}_{sx} & \boldsymbol{N}_s \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}} \\ \widehat{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}_x \\ \boldsymbol{U}_s \end{bmatrix}$$

Depending on the rank deficiency of the model different type of minimal constraints could be applied to the model:

$$\boldsymbol{C}_x^T\boldsymbol{x} + \boldsymbol{C}_s^T\boldsymbol{s} = \boldsymbol{0}$$

Least-Squares Adjustment - Spatial approach B

Deterministic approach of positions and stochastic approach of signals:

$$\boldsymbol{w} = \boldsymbol{A}\boldsymbol{x} + [\boldsymbol{G} \quad \boldsymbol{I}] \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{v} \end{bmatrix}$$

$$(\boldsymbol{A}^T \boldsymbol{M}^{-1} \boldsymbol{A})\,\hat{\boldsymbol{x}} = \boldsymbol{A}^T \boldsymbol{M}^{-1} \boldsymbol{w}$$

$$\boldsymbol{s}^T \boldsymbol{K}^{-1} \boldsymbol{s} + \boldsymbol{v}^T \boldsymbol{P} \boldsymbol{v} \rightarrow minimum \longrightarrow \hat{\boldsymbol{s}} = \boldsymbol{K}\boldsymbol{G}^T \boldsymbol{M}^{-1}(\boldsymbol{w} - \boldsymbol{A}\hat{\boldsymbol{x}})$$

$$\boldsymbol{K} = \begin{bmatrix} k_{s_1 s_1} & \cdots & k_{s_1 s_p} \\ \vdots & \ddots & \vdots \\ k_{s_p s_1} & \cdots & k_{s_p s_p} \end{bmatrix}$$

$$\boldsymbol{M} = \boldsymbol{\Sigma}_n = [\boldsymbol{G} \quad \boldsymbol{I}] \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_v \end{bmatrix} \begin{bmatrix} \boldsymbol{G}^T \\ \boldsymbol{I} \end{bmatrix} = \boldsymbol{G}\boldsymbol{K}\boldsymbol{G}^T + \boldsymbol{\Sigma}_v$$

Depending on the rank deficiency of the model different type of minimal constraints could be applied to the model:

$$\boldsymbol{C}_x^T \boldsymbol{x} + \boldsymbol{C}_s^T \boldsymbol{s} = \boldsymbol{0}$$

Links between clocks and other land instruments

$$\boldsymbol{w_x} = f(\boldsymbol{x}) + \boldsymbol{v_x}$$
$$\boldsymbol{w_s} = g(\boldsymbol{s}) + \boldsymbol{v_s}$$

$\longrightarrow$

$$\begin{bmatrix} \boldsymbol{w_x} \\ \boldsymbol{w_s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v_x} \\ \boldsymbol{v_s} \end{bmatrix}$$

$$\begin{bmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{A}^T \boldsymbol{P_x} \boldsymbol{A})^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{G}^T \boldsymbol{P_s} \boldsymbol{G})^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^T \boldsymbol{P_s} \boldsymbol{w_x} \\ \boldsymbol{G}^T \boldsymbol{P_s} \boldsymbol{w_s} \end{bmatrix}$$

Deterministic approach (common parameters):

$$\boldsymbol{w_x} = f_1(\boldsymbol{x}_1) + f_2(\boldsymbol{x}_2) + \boldsymbol{v_x}$$

$$\boldsymbol{w_s} = c(\boldsymbol{x}_2) + g(\boldsymbol{s}) + \boldsymbol{v_s}$$

$\longrightarrow$

$$\begin{bmatrix} \boldsymbol{w_x} \\ \boldsymbol{w_s} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{C} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{s} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v_x} \\ \boldsymbol{v_s} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{A}_1^T \boldsymbol{P_x} \boldsymbol{A}_1 & \boldsymbol{A}_1^T \boldsymbol{P_x} \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{A}_2^T \boldsymbol{P_x} \boldsymbol{A}_1 & \boldsymbol{A}_2^T \boldsymbol{P_x} \boldsymbol{A}_2 + \boldsymbol{C}^T \boldsymbol{P_s} \boldsymbol{C} & \boldsymbol{C}^T \boldsymbol{P_s} \boldsymbol{G} \\ \boldsymbol{0} & \boldsymbol{G}^T \boldsymbol{P_s} \boldsymbol{C} & \boldsymbol{G}^T \boldsymbol{P_s} \boldsymbol{G} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{x}}_1 \\ \hat{\boldsymbol{x}}_2 \\ \hat{\boldsymbol{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1^T \boldsymbol{P_x} \boldsymbol{w_x} \\ \boldsymbol{A}_2^T \boldsymbol{P_x} \boldsymbol{w_x} + \boldsymbol{C}^T \boldsymbol{P_s} \boldsymbol{w_s} \\ \boldsymbol{G}^T \boldsymbol{P_s} \boldsymbol{w_s} \end{bmatrix}$$

Links between clocks and other land instruments

$$\boldsymbol{w_s} = f(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{v}_s$$

$$\boldsymbol{w_{s_3}} = g(\boldsymbol{s}_3) + \boldsymbol{v}_{s_3}$$

→

$$\begin{bmatrix} \boldsymbol{w}_s \\ \boldsymbol{w}_{s_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{s}_3 \end{bmatrix} + \begin{bmatrix} \boldsymbol{v}_x \\ \boldsymbol{v}_{s_3} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{A}_1^T \boldsymbol{P}_s \boldsymbol{A}_1 & \boldsymbol{A}_1^T \boldsymbol{P}_s \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{A}_2^T \boldsymbol{P}_s \boldsymbol{A}_1 & \boldsymbol{A}_2^T \boldsymbol{P}_s \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}_3^T \boldsymbol{P}_{s_3} \boldsymbol{G}_3 \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}_1} \\ \widehat{\boldsymbol{x}_2} \\ \widehat{\boldsymbol{s}_3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1^T \boldsymbol{P}_s \boldsymbol{w}_s \\ \boldsymbol{A}_2^T \boldsymbol{P}_s \boldsymbol{w}_s \\ \boldsymbol{G}_3^T \boldsymbol{P}_{s_3} \boldsymbol{w}_{s_3} \end{bmatrix}$$

Stochastic approach between signals and position parameters:

$$\boldsymbol{w_x} = f(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{v}_s$$

$$\boldsymbol{w_{s_3}} = g(\boldsymbol{s}_3) + \boldsymbol{v}_{s_3}$$

→

$$\mathbf{K} = \begin{bmatrix} \boldsymbol{k}_{x_i x_i} & \boldsymbol{k}_{x_i s_3} \\ \boldsymbol{k}_{s_3 x_i} & \boldsymbol{k}_{s_3 s_3} \end{bmatrix}, \mathbf{i = 1, 2}$$

Links between clocks and other land instruments

$$\boldsymbol{w}_x = f(\boldsymbol{x}) + \boldsymbol{v}_x$$
$$\boldsymbol{w}_s = g(\boldsymbol{s}) + \boldsymbol{v}_s$$

$\longrightarrow$

$$\begin{bmatrix}\boldsymbol{w}_x\\ \boldsymbol{w}_s\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{G}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{s}\end{bmatrix} + \begin{bmatrix}\boldsymbol{v}_x\\ \boldsymbol{v}_s\end{bmatrix}$$

$$\begin{bmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{s}}\end{bmatrix} = \begin{bmatrix}(\boldsymbol{A}^T\boldsymbol{P}_x\boldsymbol{A})^{-1} & \boldsymbol{0}\\ \boldsymbol{0} & (\boldsymbol{G}^T\boldsymbol{P}_s\boldsymbol{G})^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}^T\boldsymbol{P}_s\boldsymbol{w}_x\\ \boldsymbol{G}^T\boldsymbol{P}_s\boldsymbol{w}_s\end{bmatrix}$$

Condition equations or external constraints between position and signal parameters:

$$\boldsymbol{w}_x = f(\boldsymbol{x}) + \boldsymbol{v}_x$$
$$\boldsymbol{w}_s = g(\boldsymbol{s}) + \boldsymbol{v}_s$$
$$\boldsymbol{Z}_x^T\boldsymbol{x} + \boldsymbol{Z}_s^T\boldsymbol{s} = \boldsymbol{d}$$

$\longrightarrow$

$$\begin{bmatrix}\boldsymbol{A}^T\boldsymbol{P}_x\boldsymbol{A} + \boldsymbol{Z}_x\boldsymbol{P}_d\boldsymbol{Z}_x^T & \boldsymbol{Z}_x\boldsymbol{P}_d\boldsymbol{Z}_s^T\\ \boldsymbol{Z}_s\boldsymbol{P}_d\boldsymbol{Z}_x^T & \boldsymbol{G}^T\boldsymbol{P}_s\boldsymbol{G} + \boldsymbol{Z}_s\boldsymbol{P}_d\boldsymbol{Z}_s^T\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{s}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}^T\boldsymbol{P}_x\boldsymbol{w}_x + \boldsymbol{Z}_x\boldsymbol{P}_d\boldsymbol{d}\\ \boldsymbol{G}^T\boldsymbol{P}_s\boldsymbol{w}_s + \boldsymbol{Z}_s\boldsymbol{P}_d\boldsymbol{d}\end{bmatrix}$$

Links between clocks and other land instruments

$$\boldsymbol{w_x} = f(\boldsymbol{x}) + \boldsymbol{v_x}$$
$$\boldsymbol{w_{s_1}} = g_1(\boldsymbol{s_1}) + \boldsymbol{v_{s_1}}$$
$$\boldsymbol{w_{s_2}} = g_2(\boldsymbol{s_2}) + \boldsymbol{v_{s_2}}$$

$\longrightarrow$

$$\begin{bmatrix} \boldsymbol{w_x} \\ \boldsymbol{w_{s_1}} \\ \boldsymbol{w_{s_2}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s_1} \\ \boldsymbol{s_2} \end{bmatrix} + \begin{bmatrix} \boldsymbol{v_x} \\ \boldsymbol{v_{s_1}} \\ \boldsymbol{v_{s_2}} \end{bmatrix}$$

$$\begin{bmatrix} \widehat{\boldsymbol{x}} \\ \widehat{\boldsymbol{s_1}} \\ \widehat{\boldsymbol{s_2}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{A}^T \boldsymbol{P}_x \boldsymbol{A})^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & (\boldsymbol{G}_1^T \boldsymbol{P}_{s_1} \boldsymbol{G}_1)^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & (\boldsymbol{G}_2^T \boldsymbol{P}_{s_2} \boldsymbol{G}_2)^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^T \boldsymbol{P}_x \boldsymbol{w_x} \\ \boldsymbol{G}_1^T \boldsymbol{P}_{s_1} \boldsymbol{w}_{s_1} \\ \boldsymbol{G}_2^T \boldsymbol{P}_{s_2} \boldsymbol{w}_{s_2} \end{bmatrix}$$

Stochastic approach of signals + external constraints between signal and position parameters:

$$\boldsymbol{w_x} = f(\boldsymbol{x}) + \boldsymbol{v_x}$$
$$\boldsymbol{w_{s_1}} = g_1(\boldsymbol{s_1}) + \boldsymbol{v_{s_1}}$$
$$\boldsymbol{w_{s_2}} = g_2(\boldsymbol{s_2}) + \boldsymbol{v_{s_2}}$$

$\longrightarrow$

$$\boldsymbol{C_x^T x} + \boldsymbol{C_{s_1}^T s_1} = \boldsymbol{d}_1$$
$$+$$
$$\boldsymbol{C_x^T x} + \boldsymbol{C_{s_2}^T s_2} = \boldsymbol{d}_2$$

$+$ $\mathrm{K} = \begin{bmatrix} \boldsymbol{k_{s_1 s_1}} & \boldsymbol{k_{s_1 s_2}} \\ \boldsymbol{k_{s_2 s_1}} & \boldsymbol{k_{s_2 s_2}} \end{bmatrix}$

Least-Squares Adjustment / Spatial-Temporal approach 1

$$\begin{bmatrix} w_{t_0} \\ w_{t_1} \\ w_{t_2} \\ \vdots \\ w_{t_n} \end{bmatrix} = \begin{bmatrix} A_{t_0} & 0 & 0 & \cdots & 0 \\ 0 & A_{t_1} & 0 & \cdots & 0 \\ 0 & 0 & A_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{t_n} \end{bmatrix} \begin{bmatrix} x_{t_0} \\ x_{t_1} \\ x_{t_2} \\ \vdots \\ x_{t_n} \end{bmatrix} + \begin{bmatrix} G_{t_0} & 0 & 0 & \cdots & 0 \\ 0 & G_{t_1} & 0 & \cdots & 0 \\ 0 & 0 & G_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{t_n} \end{bmatrix} \begin{bmatrix} s_{t_0} \\ s_{t_1} \\ s_{t_2} \\ \vdots \\ s_{t_n} \end{bmatrix} + \begin{bmatrix} v_{t_0} \\ v_{t_1} \\ v_{t_2} \\ \vdots \\ v_{t_n} \end{bmatrix}$$

Deterministic approach of positions and stochastic approach of signals:

$$\sum_{a=t_0}^{t_n} \left[v_a{}^T P_a v_a + \sum_{b=t_0}^{t_n} s_a{}^T P_{ab} s_b \right] \rightarrow minimum$$

Least-Squares Adjustment / Spatial-Temporal approach 2

$$\begin{bmatrix} w_{t_0} \\ w_{t_1} \\ w_{t_2} \\ \vdots \\ w_{t_n} \end{bmatrix} = \begin{bmatrix} A_{t_0} & 0 & 0 & \cdots & 0 \\ A_{t_1} & A_{t_1} & 0 & \cdots & 0 \\ A_{t_2} & 0 & A_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t_n} & 0 & 0 & \cdots & A_{t_n} \end{bmatrix} \begin{bmatrix} x_{t_0} \\ p_{t_1} \\ p_{t_2} \\ \vdots \\ p_{t_n} \end{bmatrix} + \begin{bmatrix} G_{t_0} & 0 & 0 & \cdots & 0 \\ G_{t_1} & G_{t_1} & 0 & \cdots & 0 \\ G_{t_2} & 0 & G_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{t_n} & 0 & 0 & \cdots & G_{t_n} \end{bmatrix} \begin{bmatrix} s_{t_0} \\ q_{t_1} \\ q_{t_2} \\ \vdots \\ q_{t_n} \end{bmatrix} + \begin{bmatrix} v_{t_0} \\ v_{t_1} \\ v_{t_2} \\ \vdots \\ v_{t_n} \end{bmatrix}$$

$p_{t_a} = x_{t_a} - x_{t_0}$ *or* $p_{t_a} = x_{t_a} - x_{t_{a-1}}$: ground displacements

$q_{t_a} = q_{t_a} - q_{t_0}$ *or* $q_{t_a} = q_{t_a} - q_{t_{a-1}}$: gravity potential variations

Stochastic approach for ground displacements and gravity potential variations:

$$p^T P_p p + q^T P_q q + \sum_{a=t_0}^{t_n} v_a{}^T P_a v_a + s_0{}^T P_{s_0} s_0 \rightarrow minimum$$

Least-Squares Adjustment / Spatial-Temporal approach 3

$$\begin{bmatrix} w_{t_0} \\ w_{t_1} \\ w_{t_2} \\ \vdots \\ w_{t_n} \end{bmatrix} = \begin{bmatrix} A_{t_0} & 0 & 0 & \cdots & 0 \\ A_{t_1} & A_{t_1} & 0 & \cdots & 0 \\ A_{t_2} & 0 & A_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t_n} & 0 & 0 & \cdots & A_{t_n} \end{bmatrix} \begin{bmatrix} x_{t_0} \\ p_{t_1} \\ p_{t_2} \\ \vdots \\ p_{t_n} \end{bmatrix} + \begin{bmatrix} G_{t_0} & 0 & 0 & \cdots & 0 \\ G_{t_1} & G_{t_1} & 0 & \cdots & 0 \\ G_{t_2} & 0 & G_{t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{t_n} & 0 & 0 & \cdots & G_{t_n} \end{bmatrix} \begin{bmatrix} s_{t_0} \\ q_{t_1} \\ q_{t_2} \\ \vdots \\ q_{t_n} \end{bmatrix} + \begin{bmatrix} v_{t_0} \\ v_{t_1} \\ v_{t_2} \\ \vdots \\ v_{t_n} \end{bmatrix}$$

$p_{t_a} = x_{t_a} - x_{t_0}$ or $p_{t_a} = x_{t_a} - x_{t_{a-1}}$: ground displacements

$q_{t_a} = q_{t_a} - q_{t_0}$ or $q_{t_a} = q_{t_a} - q_{t_{a-1}}$: gravity potential variations

Stochastic approach between ground displacements and gravity potential variations:

$$\sum_{a=t_1}^{t_n} \sum_{b=t_1}^{t_n} \left[p_a{}^T P_{p_{ab}} p_b + q_a{}^T P_{q_{ab}} q_b + p_a{}^T P_{p_a q_b} q_b + q_a{}^T P_{q_a p_b} p_b \right] + \sum_{a=t_0}^{t_n} v_a{}^T P_a v_a + s_0{}^T P_{s_0} s_0 \rightarrow minimum$$

Summary

- Two strategies have been presented for the session spatial solution:
  - Deterministic approach for positions and signals
  - Deterministic approach for positions and stochastic for signals

- Three different kind of links have been presented:
  - Deterministic approach (common parameters)
  - External constraints
  - Stochastic approach

- Three strategies have been presented for the spatial-temporal solution:
  - Deterministic approach for positions and stochastic approach for signals
  - Stochastic approach for ground displacements and gravity potential variations
  - Stochastic approach between ground displacements and gravity potential variations

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Thank you for your attention!